Self-avoiding Walks on The Hexagonal Lattice

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Contents

- Self-avoiding Walks in General
- Connective Constant on The Hexagonal Lattice

1. Generalities

Paul J. Flory

1953

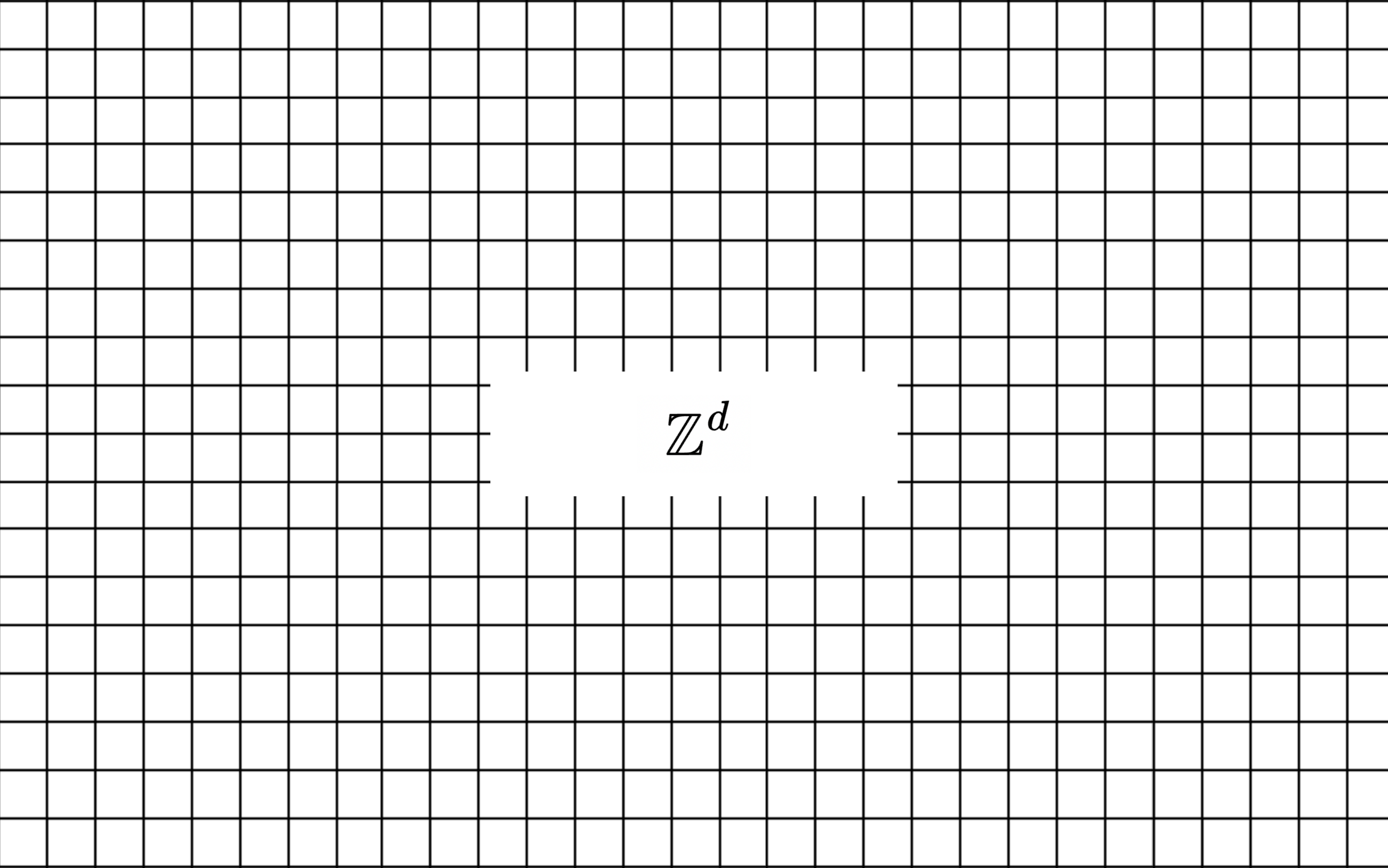
Definitions

Lattice

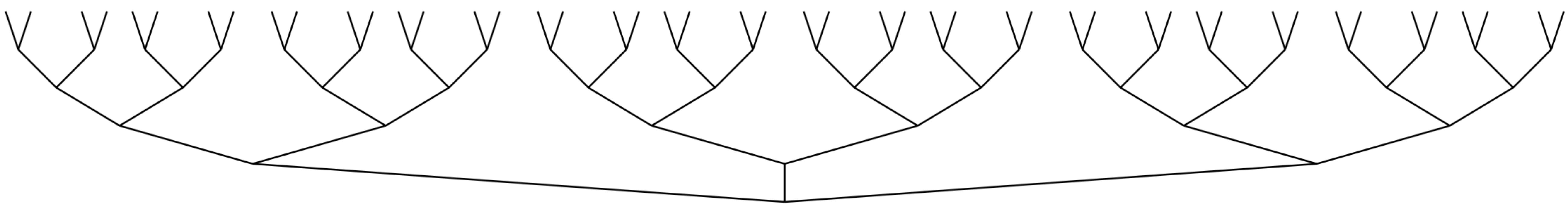
A transitive, locally-finite, infinite graph.

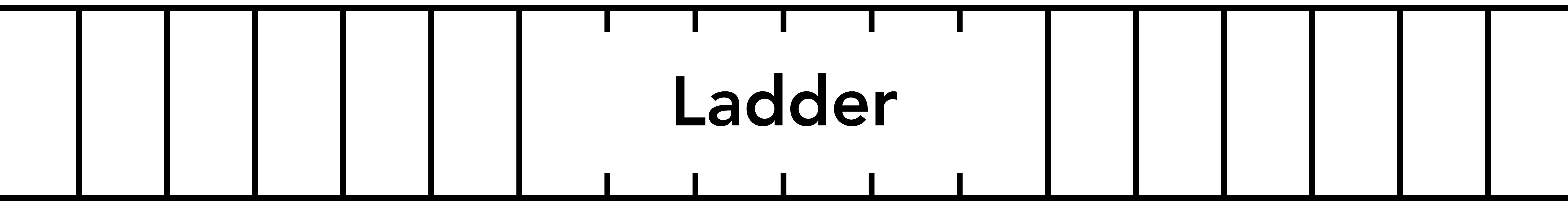
$$\mathbb{L} = (\mathbb{V}, \mathbb{E})$$

Examples


$\mathbb{Z}^d$

$\mathbb{T}_d$




Ladder

H

Walk

A sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{V}$ such that
$\{\gamma_i, \gamma_{i+1}\} \in \mathbb{E} \quad \forall \boldsymbol{i}.$

Self-avoiding Walk

A walk γ such that, if $\gamma_i = \gamma_j$, then $i = j$.

Self-avoiding Polygon

A self-avoiding walk γ such that, $\gamma_0 = \gamma_n$

and if $\gamma_i = \gamma_j$ then $i = j$ or $0 = n$.

Self-avoiding Bridge

A self-avoiding walk γ such that,

$\gamma_0 . e_1 < \gamma_i . e_1 \leq \gamma_n . e_1$ for $i = 1, \ldots, n$.

Notations

Self-avoiding Walk

SAW_n = The set of self-avoiding walks of length n.

$c_n = |SAW_n|$.

Self-avoiding Polygon

SAP_n = The set of self-avoiding polygons of length n.

$p_n = |SAP_n|$.

Self-avoiding Bridge

SAB_n = The set of self-avoiding bridges of length n.

$b_n = |SAB_n|$.

Questions

Perspective?

Combinatorics vs Probability

What is c_n?

Depends on the lattice…

$$\boldsymbol{c}_n(\mathbb{Z}^1) = \mathbf{2}.$$

$$c_n(\mathbb{T}_d) = (d+1)d^{n-1}.$$

$$c_n(\mathbb{Z}^2) = ?$$

No Formula!

$$c_n(\mathbb{H}) = ?$$

No Formula!

Numerical Estimations

$$c_{71}(\mathbb{Z}^2) \approx 4.19\cdots \times 10^{30}$$

$$c_{105}(\mathbb{H}) \approx \mathbf{5.66\cdots \times 10^{28}}$$

Exponential Growth

$$d^n \leq c_n(\mathbb{Z}^d) \leq 2d(2d-1)^{n-1}$$

What is the rate of growth?

Proposition 1.1

[Hammersley, 1954]

For any $\mathbb{L}$,

$$\lim_{n\to\infty} c_n^{1/n} = \mu_c(\mathbb{L}).$$

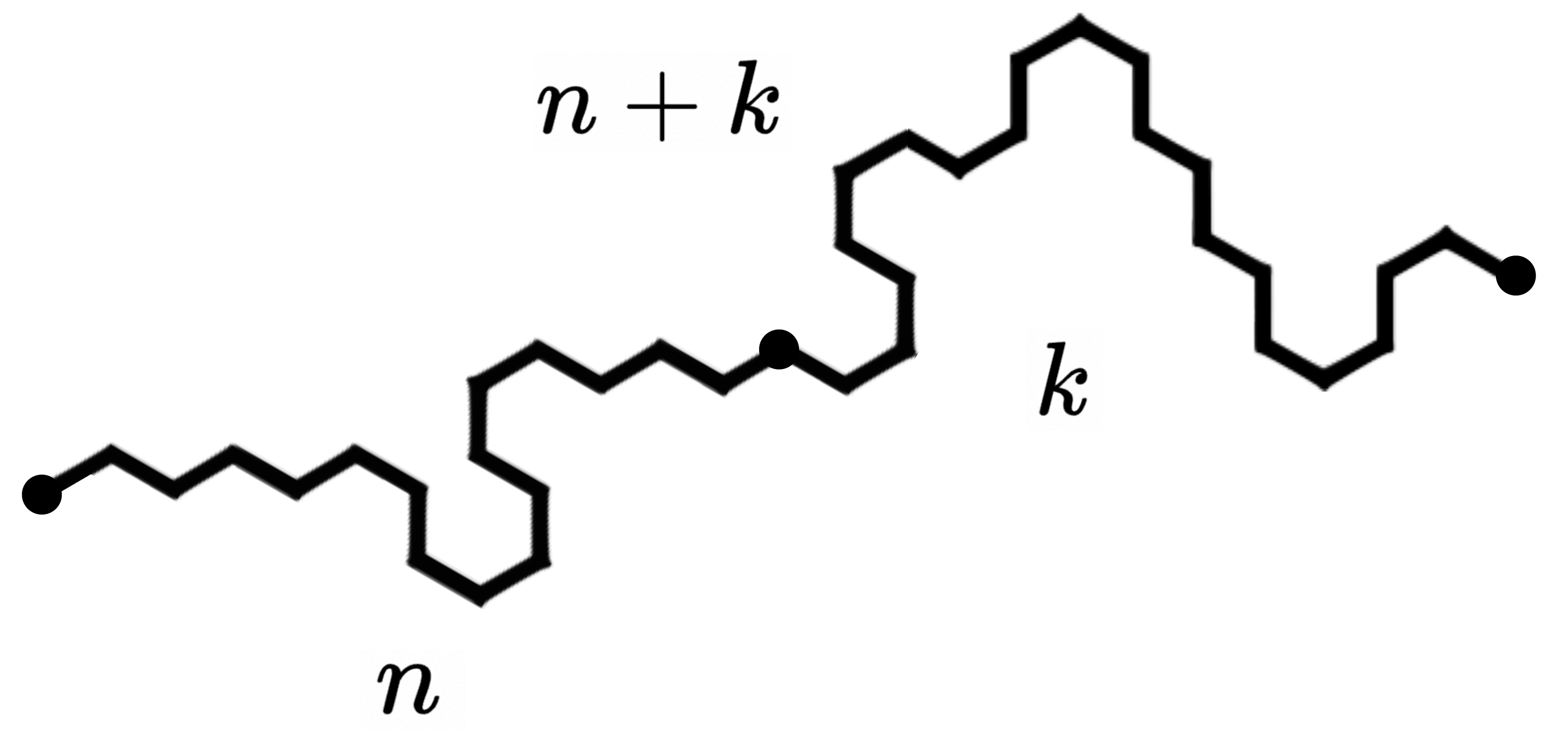
Fekete's Lemma

Let $(a_n)_{n\geq 1}$ be a subadditive sequence.

Then:

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n} \, .$$

Proof



$$\Longrightarrow \quad c_{n+k} \leq c_n . c_k$$

$\Longrightarrow \quad (\ln c_n)_{n\geq 1}$ subadditive.

$$\xrightarrow{\text{Fekete's Lemma.}} \lim_{n\to\infty} \frac{\ln c_n}{n} = \inf_{n\geq 1} \frac{\ln c_n}{n}$$

$$\Longrightarrow \quad \lim_{n\to\infty} c_n^{1/n} = \inf_{n\geq 1} c_n^{1/n} = \mu_c.$$

$\square$

Corollary 1.2

$$c_n \geq \mu_c^n, \qquad \forall n \geq 0.$$

Connective Constant

Examples

$$\mu_c(\mathbb{Z}^1) = 1$$

$$\boldsymbol{\mu}_c(\mathbb{T}_d) = \boldsymbol{d}$$

$$\mu_c(Ladder) = \frac{1 + \sqrt{5}}{2}$$

$$\mu_c(\mathbb{H}) = \sqrt{\mathbf{2} + \sqrt{\mathbf{2}}}$$

More Examples?!

Not Available!

The Only Good Example…

$$\mu_c(\mathbb{H}) = \sqrt{\mathbf{2} + \sqrt{\mathbf{2}}}$$

Theorems

Theorem 1.3

[Hammersley–Welsh, 1962]

On $\mathbb{Z}^d$, there exists a constant $c > 0$ such that:

$$c_n \leq e^{c\sqrt{n}} b_{n+1}, \quad \forall n \geq 1.$$

Corollary 1.4

$$\mu_c = \mu_b.$$

Remark

Hammersley–Welsh theorem works on the hexagonal lattice.

Proposition 1.5

Let $G(x) = \sum_{n=0}^{\infty} c_n x^n$. Then, the convergence radius of G is $\frac{1}{\mu_c}$.

2. Hexagonal Lattice

Conjecture 2.1

[Bernard Nienhuis, 1982]

$$\mu_c(\mathbb{H}) = \sqrt{2 + \sqrt{2}}.$$

Theorem 2.2

[Hugo Duminil-Copin, 2012]

$$\mu_c(\mathbb{H}) = \sqrt{2 + \sqrt{2}}.$$

Proof

Complex Plane

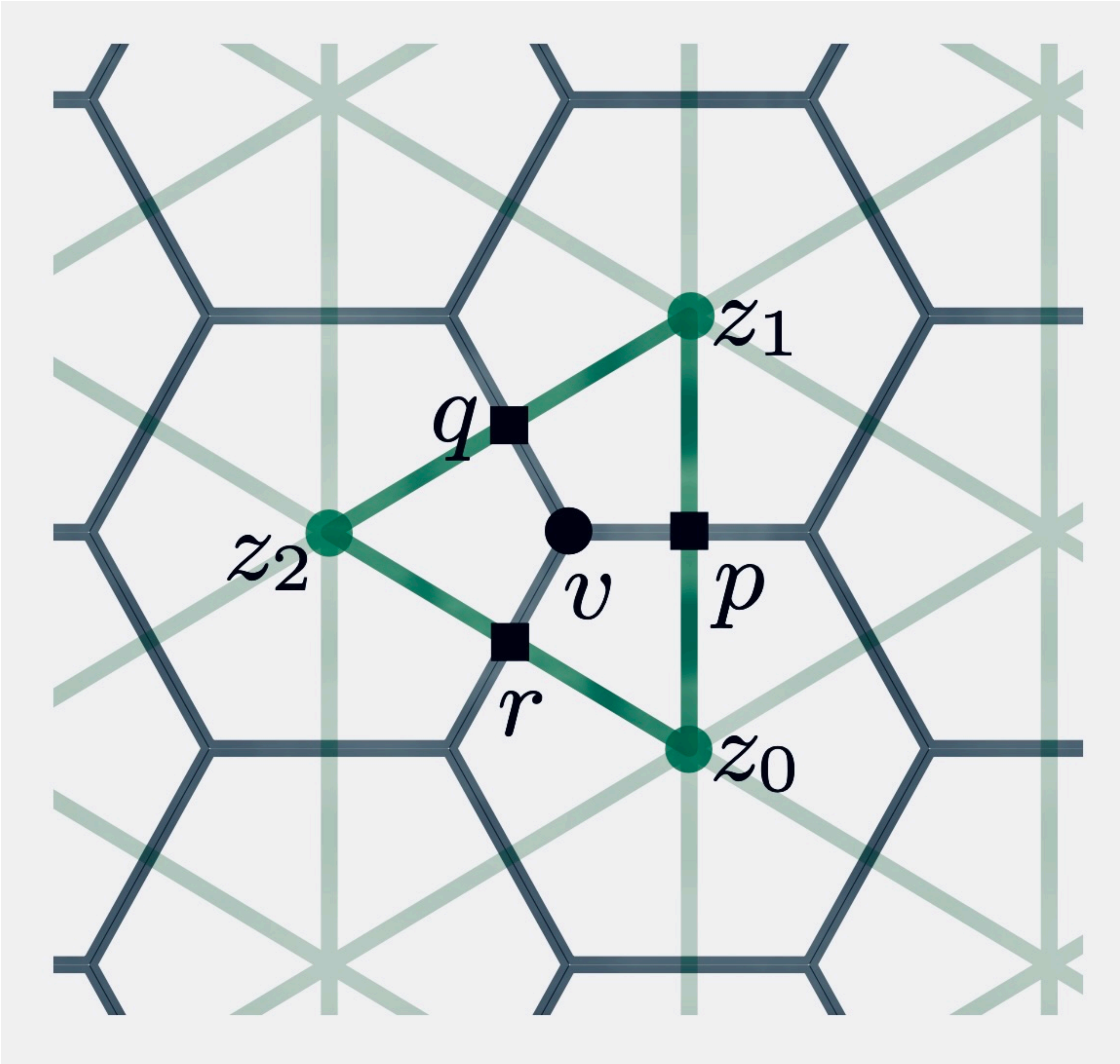
Holomorphic Functions

Discrete Holomorphic Functions

Simply Connected Domain

Discrete Simply Connected Domain

Dual Lattice

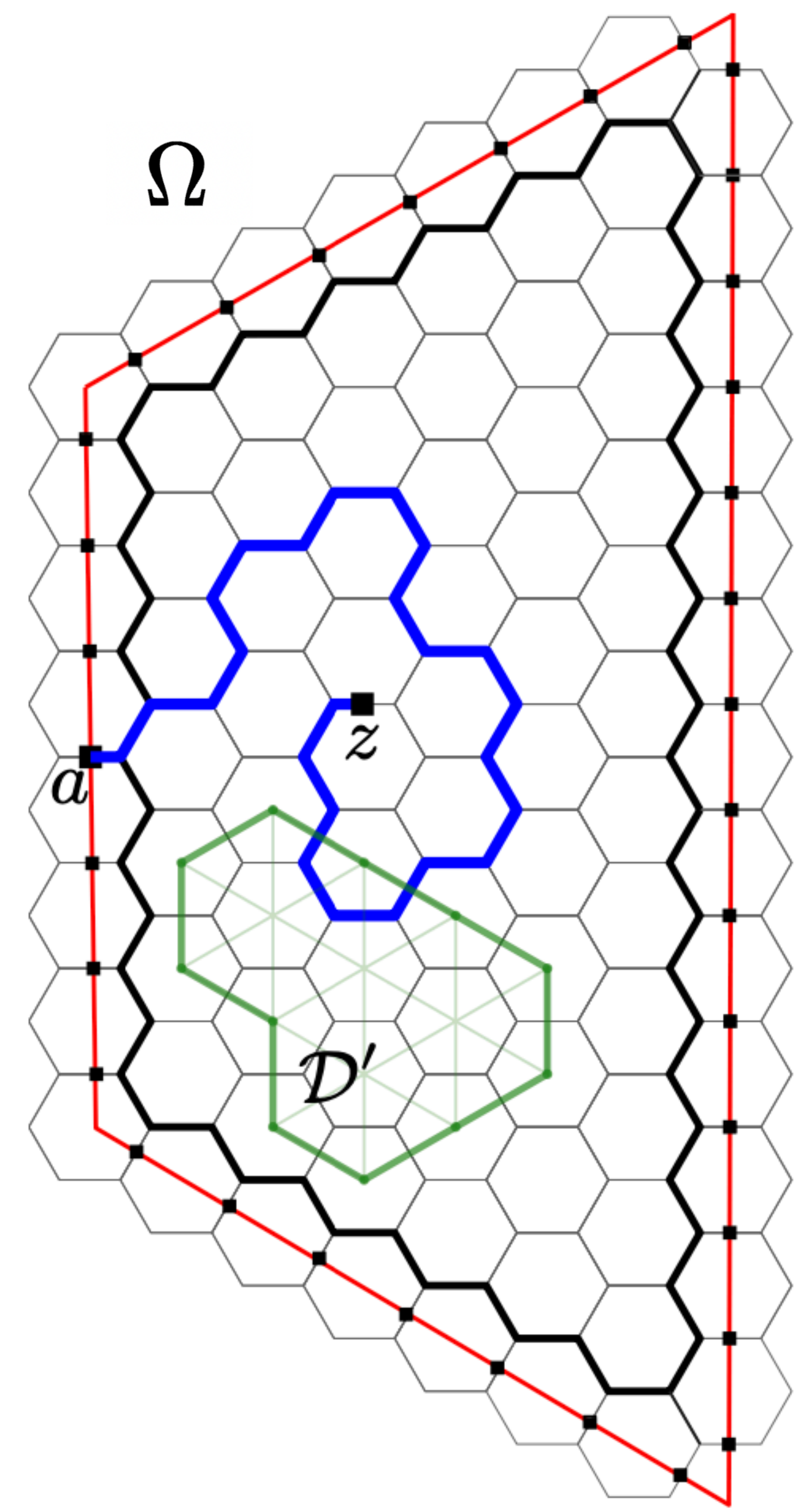


z_1
q
z_2
v
p
r
z_0

Definition

Domain Ω

The interior of a self-avoiding polygon on the dual lattice.



Ω
a
z
$\mathcal{D}'$

Fix a mid-edge $a \in \partial\Omega$

Definition

For mid-edge $z \in \Omega \cup \partial\Omega$,

$$F(z) = F_{\Omega,a,x,\sigma}(z) := \sum_{\gamma:a\rightarrow z,\gamma\subset\Omega} exp[-i\sigma W_\gamma(a,z)]x^{|\gamma|}.$$

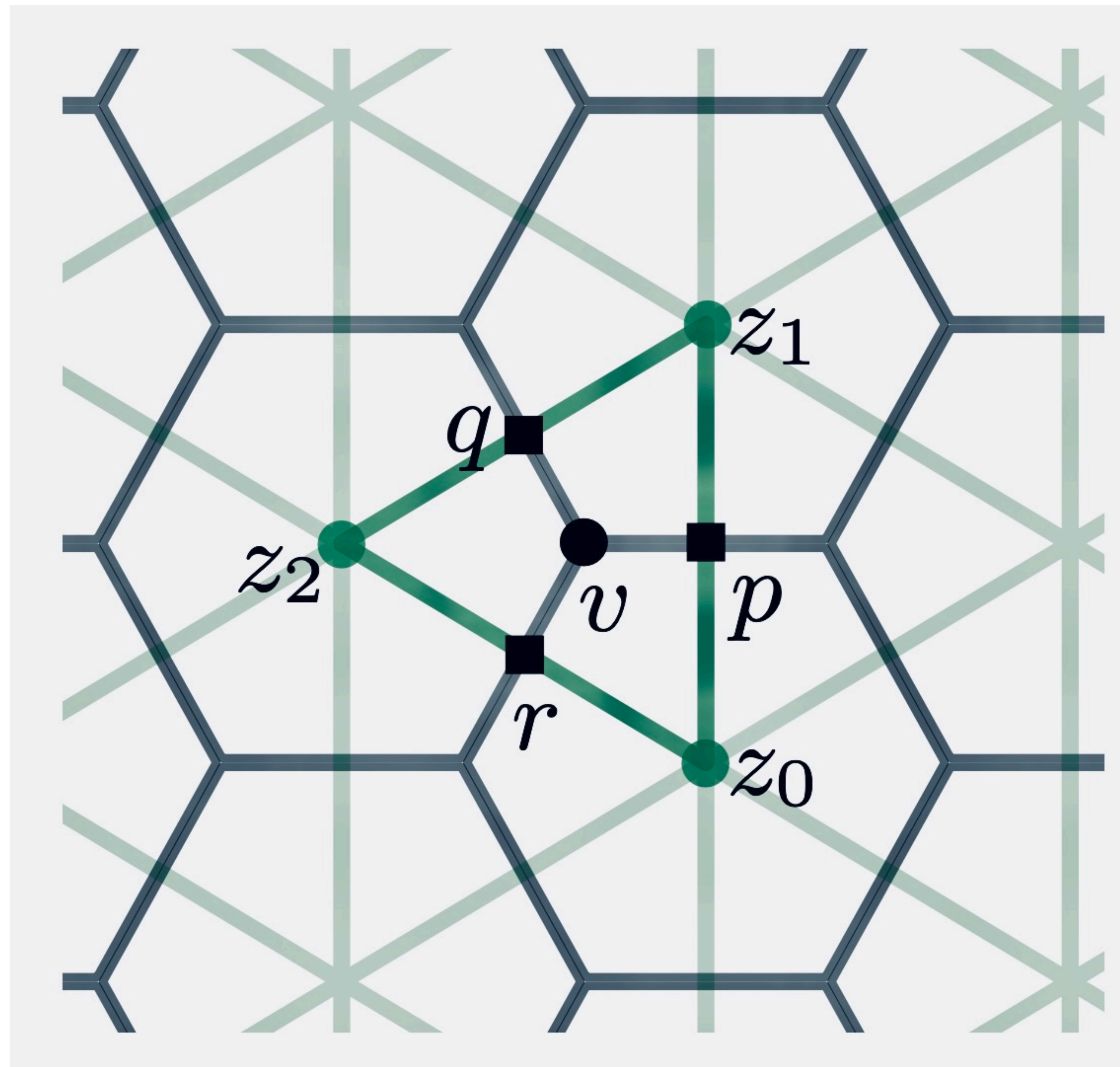
Where,

$$W_\gamma(a,z) = \frac{\pi}{3}(\#L - \#R).$$

Lemma 2.3

Let $\sigma = \frac{5}{8}$ and $x = \frac{1}{\sqrt{2+\sqrt{2}}}$. Then, for any $v \in \Omega$, and the adjacent mid-edges p, q, r:

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$



z_1
q
z_2
v
p
r
z_0

How to interpret?

Discrete Holomorphic Functions

Morera‘s Theorem

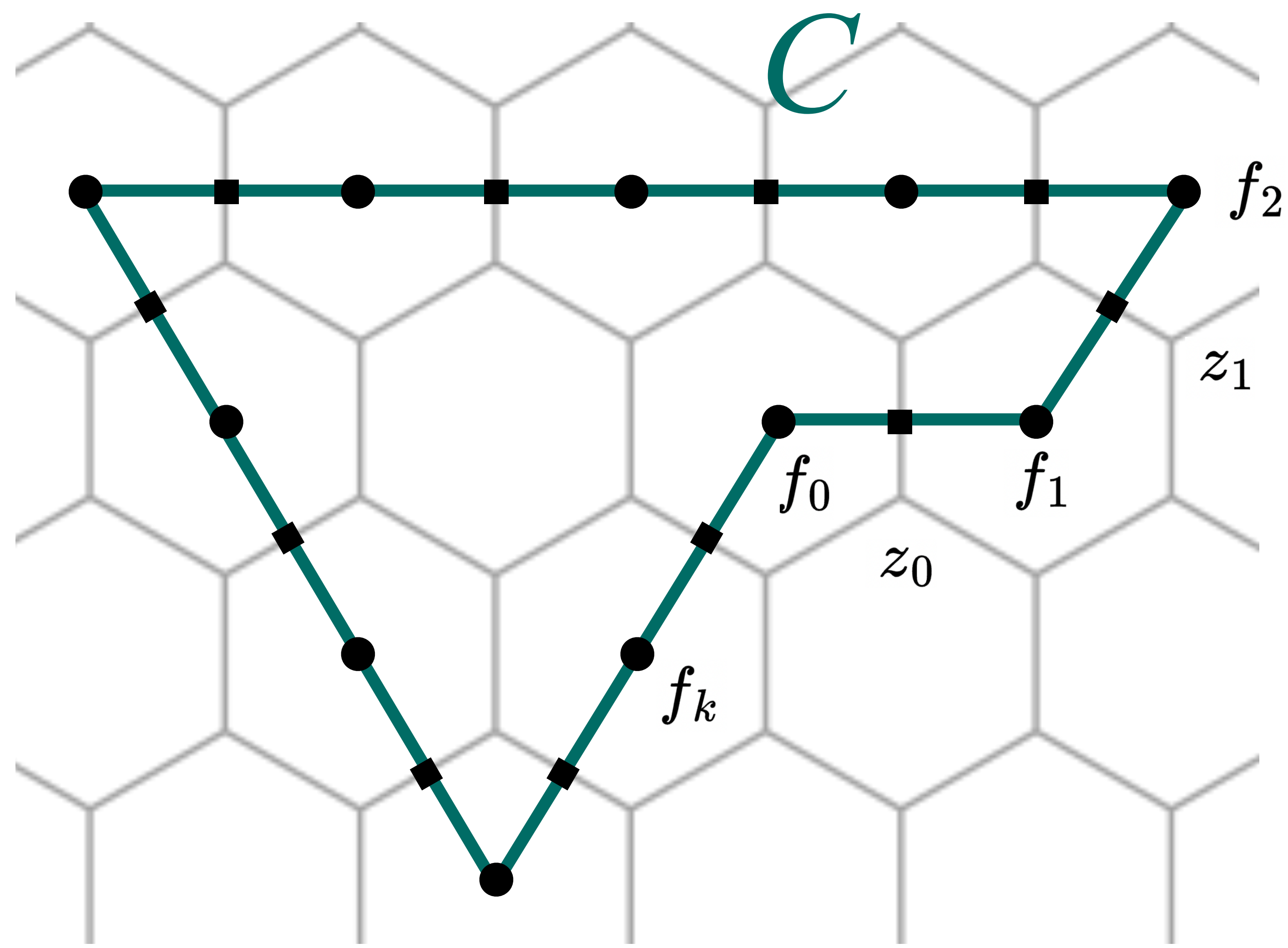
$$\oint_C f(z)dz = 0$$

True for the F!

How?

Contour C

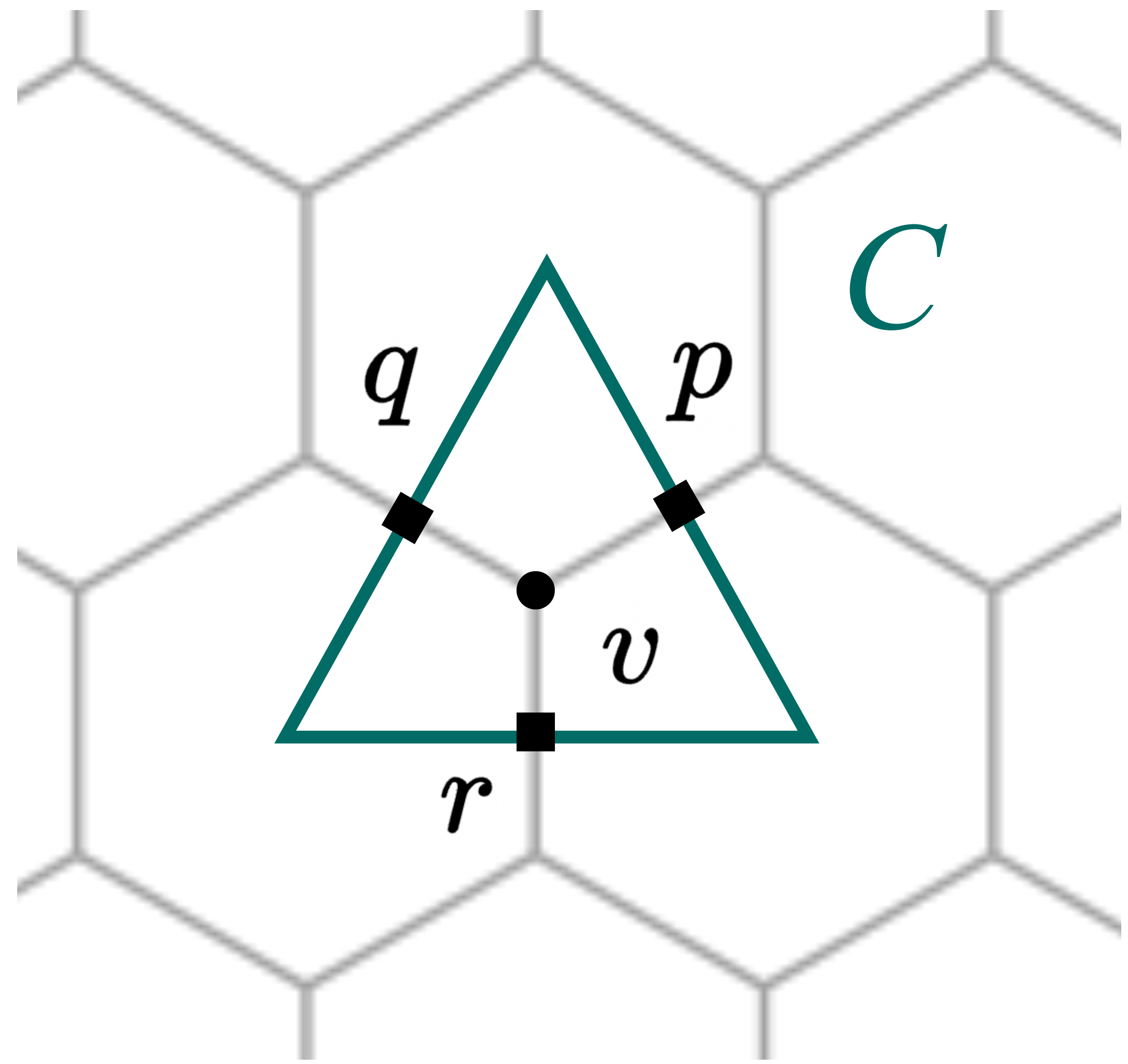
A self-avoiding polygon on the dual lattice.


C
f_2
z_1
f_0
f_1
z_0
f_k

Discrete Integration

$$\oint_C F(z)dz = \sum_{i=0}^{k}(f_{i+1} - f_i)F(z_i)$$

$$(p - v)[F(p) + e^{2i\pi/3}F(q) + e^{4i\pi/3}F(r)] = 0$$



C
q
p
v
r

$$\oint_C F(z)dz = (\ldots)[F(p) + e^{2i\pi/3}F(q) + e^{4i\pi/3}F(r)] = 0$$

$$\oint_C F(z)dz = 0 \qquad \forall C$$

Proof of Lemma 2.3

Lemma 2.3

Let $\sigma = \frac{5}{8}$ and $x = \frac{1}{\sqrt{2+\sqrt{2}}}$. Then, for any $v \in \Omega$, and the adjacent mid-edges p, q, r :

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0.$$

Definition

For mid-edge $z \in \Omega \cup \partial\Omega$,

$$F(z) = F_{\Omega,a,x,\sigma}(z) := \sum_{\gamma:a \to z, \gamma \subset \Omega} exp[-i\sigma W_\gamma(a,z)]x^{|\gamma|}.$$

Where,

$$W_\gamma(a,z) = \frac{\pi}{3}(\#L - \#R).$$

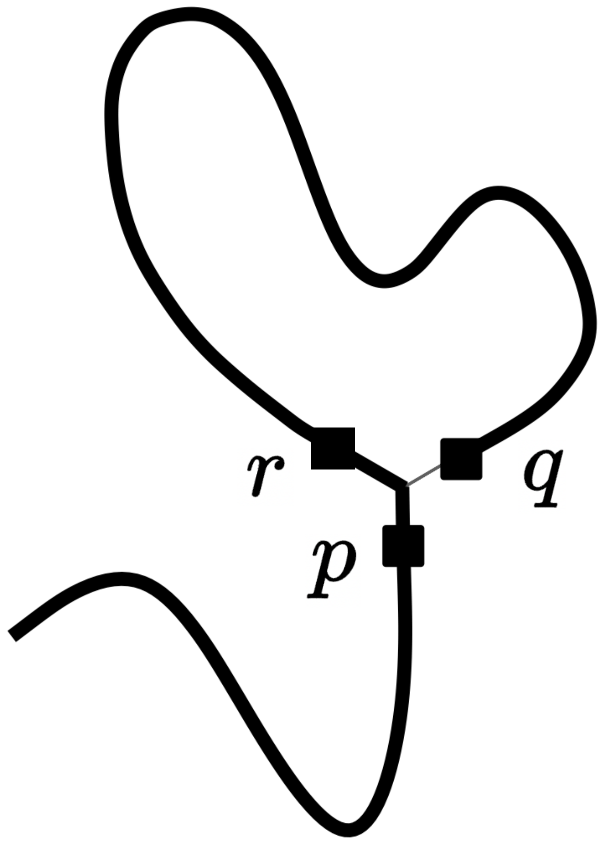
Notation

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r)=\sum_{\gamma:a\to\{p,q,r\},\gamma\subset\Omega} C(\gamma);$$

$$C(\gamma):=(\gamma_{|\gamma|}-v)e^{-i\sigma W_\gamma(a,\gamma_{|\gamma|})}x^{|\gamma|}.$$

Walks that meet all 3

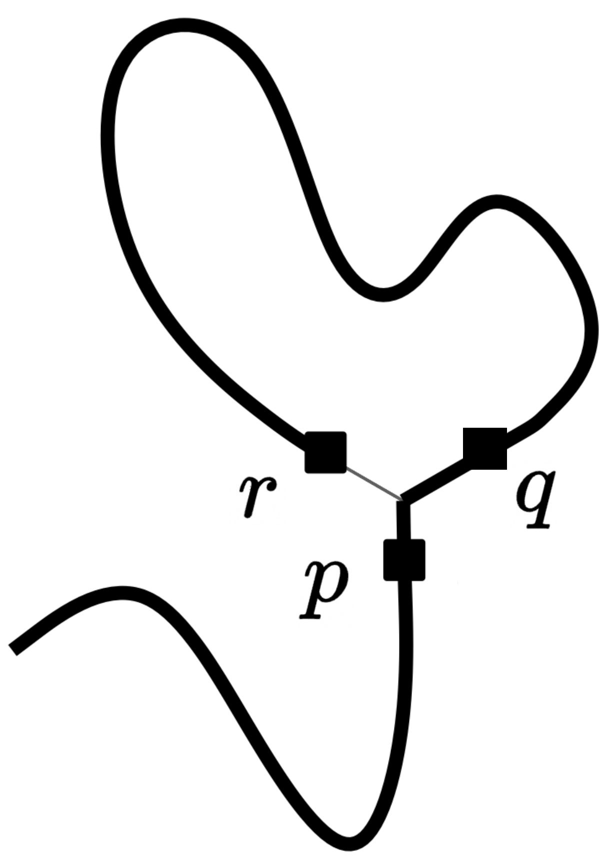
γ_1



$$C(\gamma_1) = (q - v)e^{-i\sigma W_{\gamma_1}(a,q)} x^{|\gamma_1}$$

$$= e^{i2\pi/3}(p - v)e^{-i\sigma W_{\gamma}(a,p)} e^{-i\sigma(4\pi/3)} x^{|\gamma_1|}.$$

γ_2



$$C(\gamma_2) = (r - v)e^{-i\sigma W_{\gamma_2}(a,r)} x^{|\gamma_2|}$$

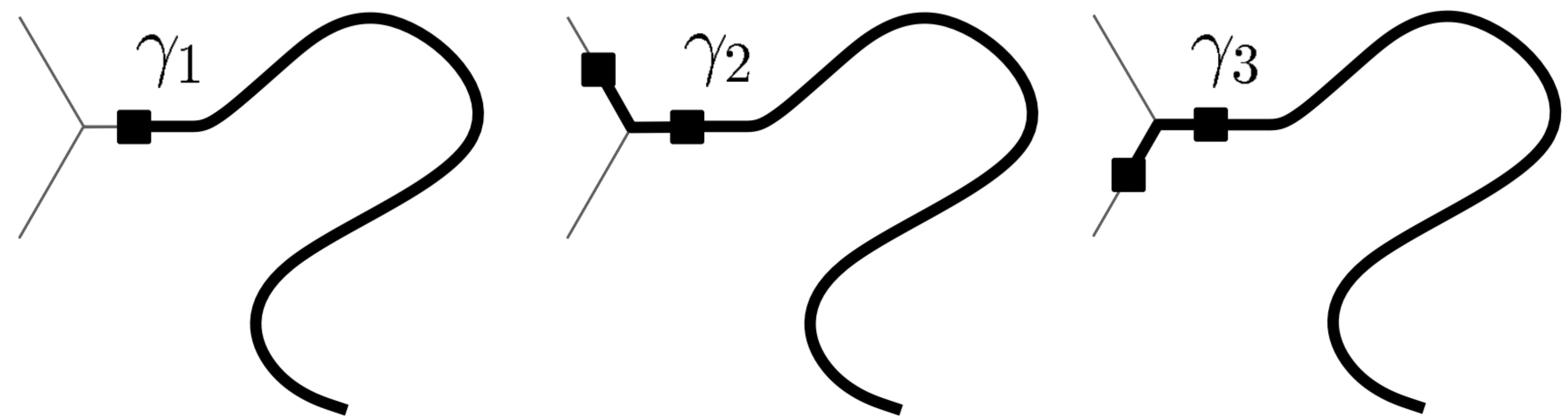
$$= e^{i4\pi/3}(p - v)e^{-i\sigma W_{\gamma}(a,p)} e^{-i\sigma(-4\pi/3)} x^{|\gamma_1|}.$$

$$C(\gamma_1) + C(\gamma_2) = (\ldots)(e^{i2\pi/3}\, e^{i\sigma 4\pi/3} + e^{-i2\pi/3}\, e^{-i\sigma 4\pi/3})$$

$$= (\ldots)(2\cos\left(\frac{2\pi}{3} + \sigma\frac{4\pi}{3}\right))$$

$$= (\ldots(2\cos\left(\frac{2\pi}{3} + \frac{5}{8}\frac{4\pi}{3}\right))) = 0.$$

Walks that meet either 2 or 1



$$C(\gamma_1) + C(\gamma_2) + C(\gamma_3)$$

$$= C(\gamma_1)(1 + xe^{-i\frac{4\pi}{3}}e^{i\sigma\frac{\pi}{3}} + xe^{-i\frac{2\pi}{3}}e^{-i\sigma\frac{\pi}{3}})$$

$$= C(\gamma_1)(1 + 2x\cos\left(\frac{7\pi}{8}\right))$$

$$= C(\gamma_1)(1 + 2\frac{1}{\sqrt{2+\sqrt{2}}}\cos\left(\frac{7\pi}{8}\right)) = 0. \qquad \square$$

Proof of Theorem 2.2

Theorem 2.2

[Hugo Duminil-Copin, 2012]

$$\mu_c(\mathbb{H}) = \sqrt{2 + \sqrt{2}}.$$

Compute explicitly?

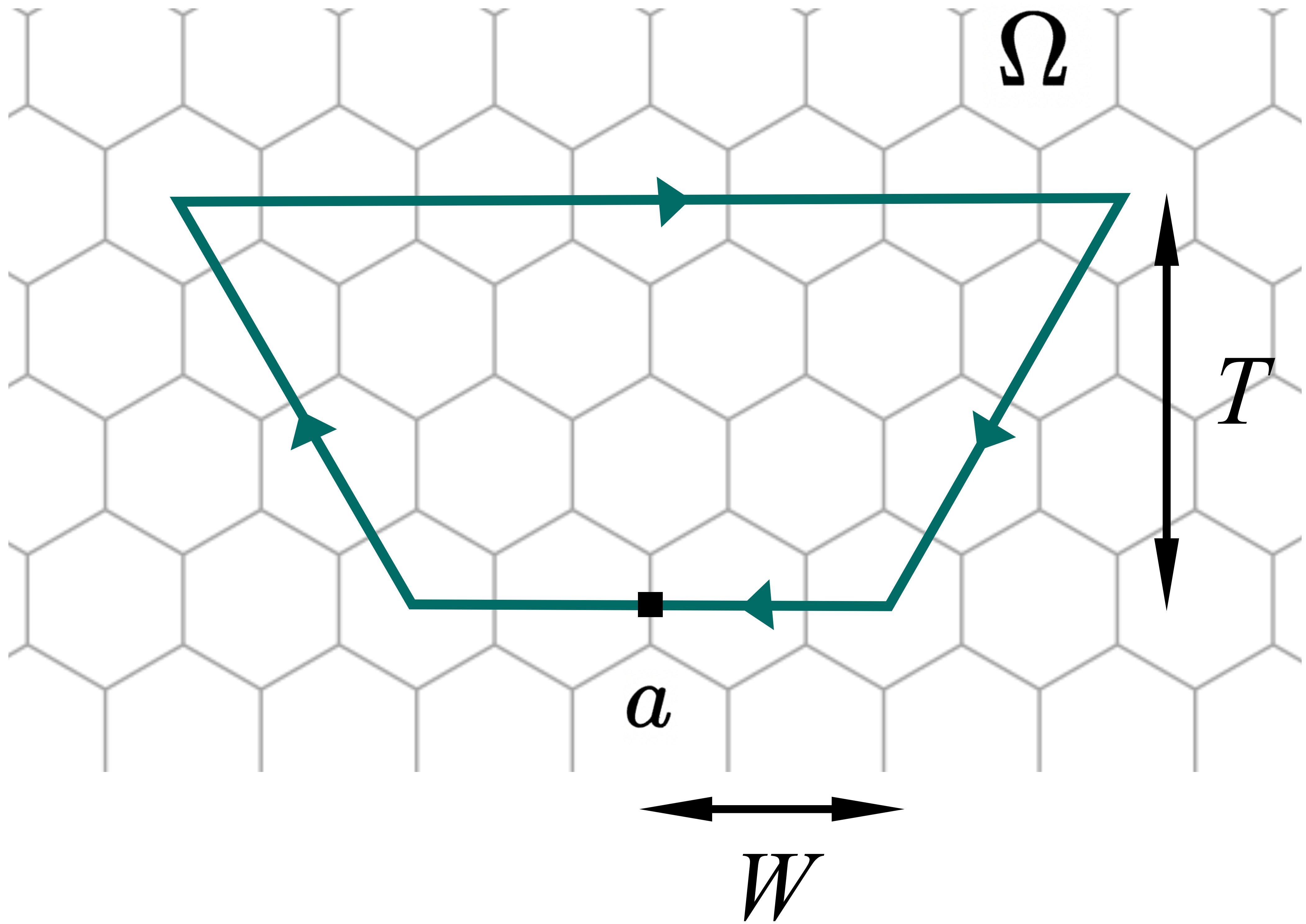
Domain

From now on,

$$\sigma = \frac{5}{8}\,, \qquad x = \frac{1}{\sqrt{2+\sqrt{2}}}\,.$$

Step 1.

$$\mu_c(\mathbb{H}) \leq \sqrt{2+\sqrt{2}}.$$



Ω
T
a
W

$$C = \partial\Omega$$

$$\oint_C F(z)dz = \int_{top} F(z)dz + \int_{bot} F(z)dz + \int_{sides} F(z)dz = 0.$$

Top

$$\boldsymbol{f_{i+1} - f_i = 1 \quad \forall i,}$$
$$\boldsymbol{W_\gamma(a, z_i) = 0 \quad \forall i.}$$

$$\implies \int_{top} F(z)dz = \sum_{\gamma:a\to top,\gamma\subset\Omega} \left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|} := B_T^W$$

Bottom

$$f_{i+1} - f_i = -1 \quad \forall i,$$

$$F(a) = 1,$$

$$\sum_{z \in bot, z \neq a} F(z) = \cos\left(\frac{5\pi}{8}\right) \sum_{\gamma : a \to bot \setminus \{a\}, \gamma \subset \Omega} \left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|}$$

$$:= \cos\left(\frac{5\pi}{8}\right) A_T^W.$$

$$\Longrightarrow \int_{bot} F(z) dz = -1 - \cos\left(\frac{5\pi}{8}\right) A_T^W.$$

Sides

$\swarrow$:

$$f_{i+1} - f_i = e^{-2\pi i/3}, \quad \forall i,$$

$$W_\gamma(a, z_i) = -2\pi/3, \forall i.$$

$$\Longrightarrow (f_{i+1} - f_i) e^{-i\sigma W_\gamma(a,z_i)} x^{|\gamma|} = e^{-\pi i/4} x^{|\gamma|}.$$

$\nwarrow$:

$$f_{j+1} - f_j = e^{2\pi i/3}, \quad \forall j,$$

$$W_\gamma(a, z_j) = 2\pi/3, \forall j.$$

$$\Longrightarrow (f_{j+1} - f_j) e^{-i\sigma W_\gamma(a,z_j)} x^{|\gamma|} = e^{\pi i/4} x^{|\gamma|}.$$

$$\Longrightarrow \quad \int_{sides} F(z) dz = \sum_{\gamma: a \to sides, \gamma \subset \Omega} x^{|\gamma|} \cos(\pi/4) \geq 0.$$

Therefore

$$B_T^W - 1 - \cos(5\pi/8) A_T^W + \int_{sides} F(z)dz = 0$$

$$\implies \quad B_T^W + \cos(3\pi/8) A_T^W + \int_{sides} F(z)dz = 1.$$

$$B_T := \lim_{W \to \infty} B_T^W .$$

$$\implies \quad B_T \leq 1, \quad \forall T.$$

$$b_n(\frac{1}{\sqrt{2+\sqrt{2}}})^n \leq \sum_{0 \leq T \leq n} B_T \leq n.$$

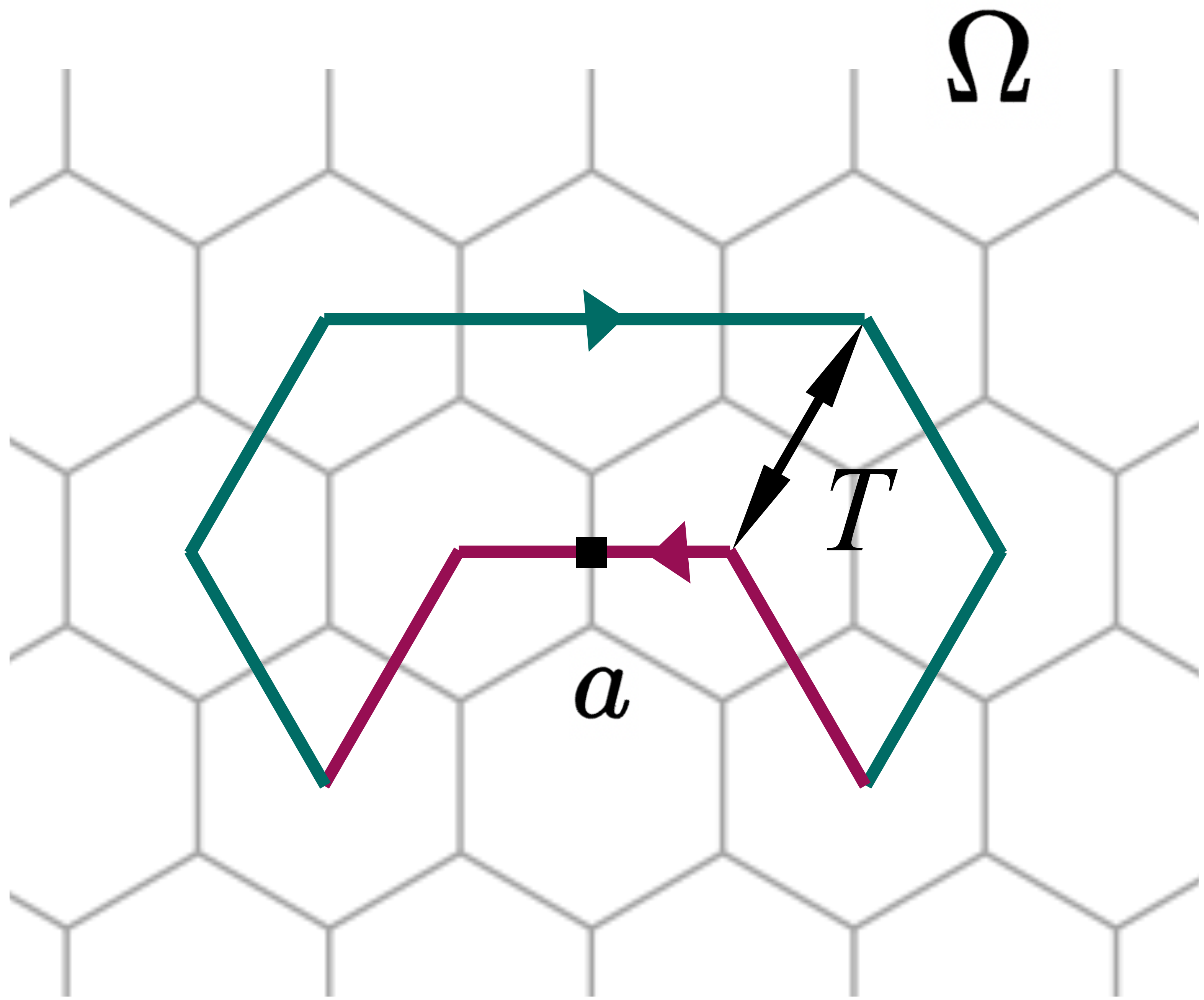
$$\implies \quad b_n \leq n\sqrt{2+\sqrt{2}}^n$$

$$\implies \quad \mu_b = \lim_{n\to\infty} b_n^{1/n} \leq \sqrt{2+\sqrt{2}} \lim_{n\to\infty} n^{1/n} = \sqrt{2+\sqrt{2}}.$$

$$\xrightarrow{\text{Hammersley-Welsh}} \mu_c \leq \sqrt{2+\sqrt{2}}.$$

Step 2.

$$\sqrt{2+\sqrt{2}} \leq \mu_c(\mathbb{H}).$$


Ω
T
a

$$C = \partial\Omega$$

$$\oint_C F(z)dz = \int_{top} F(z)dz + \int_{bot} F(z)dz = 0$$

Top

$$\cdots \implies \int_{top} F(z)dz \leq \sum_{\gamma: a \to top, \gamma \subset \Omega} \left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|}.$$

Bottom

$$F(a) = 1,$$

$$\int_{bot\setminus\{a\}} F(z)dz = (e^{2\pi i/3}e^{-i5/8(-4\pi/3)}) + (e^{-2\pi i/3}e^{-i5/8(4\pi/3)})$$

$$= 2\cos(2\pi/3 + 5\pi/6)$$

$$= 2\cos(3\pi/2) = 0.$$

$$\implies \int_{bot} F(z)dz = -1.$$

Therefore

$$\int_C F(z)dz = \int_{top} F(z)dz - 1 = 0$$

$$\implies \quad \int_{top} F(z)dz = 1$$

$$\implies \quad \sum_{\gamma:a\to top,\gamma\subset\Omega} \left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|} \geq 1.$$

$$G(x) = \sum_{n=1}^{\infty} c_n x^n$$

$$\implies \quad G\left(\frac{1}{\sqrt{2+\sqrt{2}}}\right) \geq \sum_{T=1}^{\infty} \sum_{\gamma:1 \to top, \gamma \subset \Omega} \left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|}$$

$$\geq \sum_{T=1}^{\infty} 1 = \infty.$$

$$\xrightarrow{\text{Proposition 1.5}} \frac{1}{\sqrt{2+\sqrt{2}}} \geq \frac{1}{\mu_c}$$

$$\implies \quad \sqrt{2+\sqrt{2}} \leq \mu_c(\mathbb{H}).$$

$\square$

To Sum UP

Thanks

References

[1] R. Bauerschmidt, H. Duminil-Copin, J. Goodman, and G. Slade, *Lectures on self-avoiding walks*, Probability and statistical physics in two and more dimensions, Clay Mathematics Proceedings, vol. 15, American Mathematical Society, 2012, pp. 395–467.